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Iwanami Studies in Advanced Mathematics

Hilbert Spaces of Analytic Functions (2)

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This paper is a translation of the above book from Chapter 1 Section 3 to Section 5.

Section 3　Normal Orthogonal Basis

A countable subset $\{e_n\}$ of a Hilbert space $\mathcal{H}$ is called an orthonormal system of $\mathcal{H}$ if it satisfies the condition $\langle e_n, e_m\rangle=\delta_{nm}$, where δ_{nm} is the Kronecker delta. When the smallest closed subspace containing the orthonormal system $\{e_n\}$ denoted as

$$[\{e_n\}]=\{\text{closure of all linear combinations of } \{e_n\}\}$$

is equal to $\mathcal{H}$, then $\{e_n\}$ is called an orthonormal basis of $\mathcal{H}$.

Proposition I-1.3.1. Let $\{e_n\}$ be an orthonormal system of $\mathcal{H}$, and F be a closed subspace of $\mathcal{H}$ generated by $\{e_n\}$. For any $g\in\mathcal{H}$, if

$$f=\sum_{n=1}^{\infty}\langle g, e_n\rangle e_n,$$

then $f\in F$ and f is the orthogonal projection of g onto F.

Proof Let $F_N=[e_1, e_2, \cdots, e_N]$. For $g\in\mathcal{H}$, let

$$f_N=\sum_{n=1}^{N}\langle g, e_n\rangle e_n.$$

Then f_N is the orthogonal projection of g onto F_N. If we can show that $\mathrm{dist}(g, F_N)\to\mathrm{dist}(g, F)$ $(N\to\infty)$, then $\|g-f_N\|\to\mathrm{dist}(g, F)$. Since $\|f-f_N\|\to 0 (N\to\infty)$, it follows that $\|g-f\|=\mathrm{dist}(g, F)$. By the proof of Theorem I-1.2.2, f is the orthogonal projection of g onto F.

We show that $\mathrm{dist}(g, F_N) \to \mathrm{dist}(g, F) = \delta$. For any N, $\|g - f_N\| = \mathrm{dist}(g, F_N)$, Since $f_N \in F_N \subset F_{N+1} \subset F$, it follows that

$$\|g - f_N\| \geq \|g - f_{N+1}\| \geq \delta.$$

For any $\varepsilon > 0$, there exists $h \in F$ such that

$$\delta + \varepsilon > \|g - h\| \geq \delta.$$

For this h, there exists N_1 and $h' \in F_{N_1}$ such that $\|h - h'\| < \varepsilon$. Since $\delta + 2\varepsilon > \|g - h'\| \geq \delta$, it follows that

$$\delta + 2\varepsilon > \mathrm{dist}(g, F_{N_1}) = \|g - f_{N_1}\| \geq \delta.$$

Therefore $\lim_{N\to\infty} \mathrm{dist}(g, F_N) = \mathrm{dist}(g, F)$. (end of proof)

Corollary I-1.3.2. (Bessel's inequality) If $\{e_n\}$ is an orthonormal system of $\mathcal{H}$, then

$$\sum_{n=1}^{\infty} |\langle g, e_n\rangle|^2 \leq \|g\|^2 \quad (g \in \mathcal{H}).$$

The equality holds when $\{e_n\}$ is a basis.

Proof We use the same symbols as in Proposition I-1.3.1. Then $\|f - f_N\| \to 0 (N \to \infty)$ and $\|f_N\|^2 = \sum_{n=1}^{N} |\langle g, e_n\rangle|^2$. Since f is the orthogonal projection of g onto F, we have $\|f\| \leq \|g\|$. Since $\|f_N\|^2$ is increasing, we have $\|f_N\|^2 \leq \|g\|^2$. If we set $f = g$, then the latter half follows. (End of proof)

Section 4 Bounded Linear Functionals

If $\mathcal{H}$ is a Hilbert space and $\mathcal{H}^*$ is the set of bounded linear functionals on $\mathcal{H}$, then $\mathcal{H}^*$ is a normed vector space. For $g \in \mathcal{H}$, if

$$\phi_g(f) = \langle f, g\rangle \quad (f \in \mathcal{H})$$

then, by Cauchy-Schwarz inequality, $\phi_g \in \mathcal{H}^*$. In this case, $\|\phi_g\| = \|g\|$. For $\alpha, \beta \in \mathbb{C}$ and $g, h \in \mathcal{H}$,

$$\phi_{\alpha g + \beta h} = \bar{\alpha}\phi_g + \bar{\beta}\phi_h.$$

The following Theorem I-1.4.1 is called the Riesz representation theorem, and shows that $\mathcal{H}$ is identified with $\mathcal{H}^*$.

Theorem I-1.4.1. (Riesz representation theorem) For any bounded linear functional ϕ on $\mathcal{H}$, there exists a unique vector $g\in\mathcal{H}$ such that

$$\phi(f)=\langle f, g\rangle \quad (f\in\mathcal{H}).$$

Proof Let Ker $\phi=\{f\in\mathcal{H}: \phi(f)=0\}$. Since ϕ is bounded, Ker ϕ is a closed subspace of $\mathcal{H}$. If Ker $\phi=\mathcal{H}$, then $\phi(f)=\langle f, 0\rangle (f\in\mathcal{H})$, so the theorem is true for this case. If Ker $\phi\neq\mathcal{H}$, then there exists h of norm 1 such that $h\perp$Ker ϕ. Since $h\notin$Ker ϕ, $\phi(h)\neq 0$. For $f\in\mathcal{H}$, $f-(\phi(f)/\phi(h))h\in$Ker ϕ. Therefore, for $f\in\mathcal{H}$,

$$\begin{aligned}\phi(f)=\phi(f)\langle h, h\rangle&=\left\langle\frac{\phi(f)}{\phi(h)}h, \overline{\phi(h)}h\right\rangle\\ &=\left\langle f, \overline{\phi(h)}h\right\rangle+\left\langle\frac{\phi(f)}{\phi(h)}h-f, \overline{\phi(h)}h\right\rangle\\ &=\left\langle f, \overline{\phi(h)}h\right\rangle.\end{aligned}$$

Therefore, if $g=\overline{\phi(h)}h$, then $\phi(f)=\langle f, g\rangle$.

If $\langle f, g_1\rangle=\langle f, g_2\rangle$ for all $f\in\mathcal{H}$, then in particular, for $f=g_1-g_2$, $\langle g_1-g_2, g_1-g_2\rangle=0$, so $g_1=g_2$, which proves uniqueness. (End of proof)

Theorem I-1.4.2. The unit ball of $\mathcal{H}$ is weakly compact.

Proof By Theorem I-1.4.1 and the remarks, $\mathcal{H}^*=\mathcal{H}$, so this is the result of Theorem II-1.1.9. (End of proof)

Corollary I-1.4.3. Let M be a closed subspace of $\mathcal{H}$ and $\phi\in\mathcal{H}^*$. If ϕ is not zero on M, then there is a unique solution $F=F_M\in M$ to the extremal problem

$$\sup\{\mathrm{Re}\,\phi(f): f\in M, \|f\|\leq 1\}$$

Proof If

$$\alpha=\sup\{\mathrm{Re}\,\phi(f): f\in M, \|f\|\leq 1\}$$

then there exists $\{f_n\}\subset M$ such that $\|f_n\|\leq 1$ and Re $\phi(f_n)\to\alpha$. By Theorem I-1.4.2, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$, and f_{n_j} converges to $F\in\mathcal{H}$, which is a weak topology. By Theorem II-1.1.11, M is closed in the weak topology of $\mathcal{H}$, so $F\in M$ and Re $\phi(F)=\alpha$.

We shall show that F is a unique solution. If G is another solution to this extremal problem, then $tF+(1-t)G$ also solves the same extremal problem for any $0<t<1$.

In this case, since ϕ is not zero, for any t

$$\|tF+(1-t)G\|=1=\|tF\|+\|(1-t)G\|$$

Therefore, $\langle tF, (1-t)G\rangle=\|tF\|\cdot\|(1-t)G\|$.

By Lemma I-1.1.2, $(1-t)G$ indicates a positive scalar multiple of tF, so $G=F$. (End of proof)

Theorem I-1.4.4. If $\mathscr{H}$ is separable, then the unit ball of $\mathscr{H}$ endowed with the weak topology is metrizable.

Proof Let $\mathscr{H}_1$ be the unit ball of $\mathscr{H}$, and $\{e_n\}$ be an orthonormal basis of $\mathscr{H}$. For $f, g\in\mathscr{H}_1$, if

$$d(f, g)=\sum_{j=1}^{\infty}2^{-(j+1)}|\langle f-g, e_j\rangle|$$

then d is the distance of $\mathscr{H}_1$. If $f_n\in\mathscr{H}_1$ weakly converges to $f\in\mathscr{H}_1$, then $d(f_n, f)\to 0$, since $\langle f_n-f, e_j\rangle\to 0$ and $|\langle f_n-f, e_j\rangle|\leq 2$, $d(f_n, f)\to 0$. Conversely, if $d(f_n, f)\to 0$, then $\langle f_n-f, e_j\rangle\to 0$. However, for any $x\in\mathscr{H}_1$, we can write $x=\sum_{j=1}^{\infty}\alpha_j e_j$ by Proposition I-1.3.1. Therefore, using $\|f_n-f\|\leq 2$ and the Cauchy-Schwarz inequality (I-1.1.2),

$$|\langle f_n-f, x\rangle|\leq 2\left\|x-\sum_{j=0}^{\ell}\alpha_j e_j\right\|+\left|\left\langle f_n-f, \sum_{j=0}^{\ell}\alpha_j e_j\right\rangle\right|.$$

This implies that $\langle f_n-f, x\rangle\to 0$. (End of proof)

Section 5　Bounded Linear Operator

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and T be a linear operator from $\mathscr{H}$ to $\mathscr{K}$.

T is said to be bounded if there exists a positive constant γ such that

$$\|Tf\|_{\mathscr{K}}\leq\gamma\|f\|_{\mathscr{H}}\quad (f\in\mathscr{H}).$$

$\|T\|$ represents the lower bound of such γ.

The set of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$ is written by $\mathscr{B}(\mathscr{H}, \mathscr{K})$. If $\mathscr{H}=\mathscr{K}$, then $\mathscr{B}(\mathscr{H})$ in short. It is well known that T is bounded if and only if T is continuous.

The set of all $\lambda\in\mathbb{C}$ for which $T-\lambda I$ is not invertible for $T\in\mathscr{B}(\mathscr{H})$ is denoted by $\sigma(T)$, where I is the identity operator on $\mathscr{H}$. $\sigma(T)$ is called the spectrum of T, and is known to be a compact set of $\mathbb{C}$ that is not the empty set.

Lemma I-1.5.1. For any $g\in\mathscr{K}$, if there exists a bounded linear functional Φ_g on $\mathscr{H}$ such that

$$|\Phi_g(f)| \leq \gamma \|f\|_{\mathcal{H}} \|g\|_{\mathcal{K}} \quad (f \in \mathcal{H})$$

and there exists a positive constant γ such that

$$|\Phi_g(f)| \leq \gamma \|f\|_{\mathcal{H}} \|g\|_{\mathcal{K}} \quad (f \in \mathcal{H})$$

then there exists a bounded linear operator S from $\mathcal{K}$ to $\mathcal{H}$ such that

$$\Phi_g(f) = \langle f, Sg \rangle_{\mathcal{H}} \quad (g \in \mathcal{K})$$

Proof By the Riesz representation theorem I-1.4.1, there exists a unique $h \in \mathcal{H}$ such that $\Phi_g(f) = \langle f, h \rangle_{\mathcal{H}}$, so let $Sg = h$. In this case, it is clear that $\|S\| \leq \gamma$. (End of proof)

Proposition I-1.5.2. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, there exists a unique $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\langle Tf, g \rangle_{\mathcal{K}} = \langle f, Sg \rangle_{\mathcal{H}} \quad (f \in \mathcal{H}, g \in \mathcal{K}).$$

In this case, we write $S = T^*$ and call it the conjugate operator of T.

Proof For $g \in \mathcal{K}$,

$$\Phi_g(f) = \langle Tf, g \rangle_{\mathcal{K}} \quad (f \in \mathcal{H})$$

By the Cauchy-Schwarz inequality, Φ_g satisfies Lemma I-1.5.1, so there exists a bounded linear operator S from $\mathcal{K}$ to $\mathcal{H}$ such that

$$\langle Tf, g \rangle_{\mathcal{K}} = \Phi_g(f) = \langle f, Sg \rangle_{\mathcal{H}} \quad (f \in \mathcal{H}, g \in \mathcal{K}).$$

We shall show the uniqueness. If there exists another $S' \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\langle Tf, g \rangle_{\mathcal{K}} = \langle f, S'g \rangle_{\mathcal{H}}$ $(f \in \mathcal{H}, g \in \mathcal{K})$, then $\langle f, Sg - S'g \rangle_{\mathcal{H}} = 0$ $(f \in \mathcal{H})$. Therefore, $(S - S')g = 0$ $(g \in \mathcal{K})$, so $S = S'$. (End of proof)

Corollary I-1.5.3. If $T \in \mathcal{B}(\mathcal{H})$, then $\mathcal{H} = \operatorname{Ker} T^* \oplus [\operatorname{Ran} T]$, where $\operatorname{Ker} T^* = \{f \in \mathcal{H} : T^* f = 0\}$ and $\operatorname{Ran} T = T\mathcal{H}$.

Proof This is clear from the above proposition.

Theorem I-1.5.4. (Schur's theorem) Let $T = (a_{ij})$ be the representation matrix of the linear operator T with respect to an orthonormal basis on ℓ^2. If there exist $0 < \gamma < \infty$ and

$0<h_j<\infty (1\leq j<\infty)$ such that

$$\sum_{j=1}^{\infty}|a_{ij}|h_j\leq\gamma h_i \quad (i\geq 1),$$

$$\sum_{i=1}^{\infty}|a_{ij}|h_i\leq\gamma h_j \quad (j\geq 1),$$

then T is bounded on ℓ^2 and $\|T\|\leq\gamma$.

Proof Let $(f_j)\in\ell^2$ and $1\leq n<\infty$. Let fix $1\leq i<\infty$. Then

$$\begin{aligned}\left|\sum_{j=1}^{n}a_{ij}f_j\right| &\leq \sum_{j=1}^{n}|a_{ij}|h_j^{1/2}h_j^{-1/2}|f_j| \\ &\leq\left(\sum_{j=1}^{n}|a_{ij}|h_j\right)^{1/2}\left(\sum_{j=1}^{n}|a_{ij}|h_j^{-1}|f_j|^2\right)^{1/2} \\ &\leq\sqrt{\gamma}\,h_i^{1/2}\left(\sum_{j=1}^{n}|a_{ij}|h_j^{-1}|f_j|^2\right)^{1/2}.\end{aligned}$$

Therefore, from the assumption,

$$\begin{aligned}\sum_{i=1}^{n}\left|\sum_{j=1}^{n}a_{ij}f_j\right|^2 &\leq\gamma\sum_{i=1}^{n}\left(h_i\sum_{j=1}^{n}|a_{ij}|h_j^{-1}|f_j|^2\right) \\ &=\gamma\sum_{j=1}^{n}h_j^{-1}|f_j|^2\left(\sum_{i=1}^{n}|a_{ij}|h_i\right) \\ &\leq\gamma\sum_{j=1}^{n}h_j^{-1}|f_j|^2\times\gamma h_j=\gamma^2\sum_{j=1}^{n}|f_j|^2.\end{aligned}$$

(End of Proof)