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ガウス型定常確率過程における
過去・現在・未来

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1．序　　文

平均が 0 であるような確率変数の集合 G の任意の元 $X_1, \ldots, X_n$ について，その 1 次結合 Y が Gauss 分布：

$$P(a \leq Y < b) = \frac{1}{\sqrt{2\pi V}} \int_a^b e^{-y^2/2V} dy$$

を満足するとき，G は Gauss 属であるという。（cf. H. Dym and H.P. McKean [17]）ただし，$V = E(Y^2)$ である。このとき，

$$\|Y\| = \{E(Y^2)\}^{1/2}$$

として G に Hilbert 空間の構造を導入し，そのようにしてできた Hilbert 空間を同じ記号 G で表す。G はそれ自身 Gauss 属であるし，その部分空間達も Gauss 属である。G の 2 つの部分空間 A と B の間の独立性を測る量として，A と B がなす角の cosine:

$$c_1 = \text{supremum of } E(XY) \text{ for } X \in A \text{ and } Y \in B \text{ of length} \leq 1$$

がある。Gauss 属 G の任意の元が時間のパラメータ t により $X(t)$ と表されているとする。もし

$$P(\cap_{i=1}^{n} (a_i \leq X(t_i + T) < b_i))$$

が T に依存しないならば，G はガウス型定常確率過程と呼ばれる。以下では，G のスペクトル関数が W である場合を考える。

$L^2(W)$ から $H^2(W)$ への射影作用素を P で表し，$Q = I - P$ と定める。このとき，

$$P\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n,$$

$$Q\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{-1} a_n z^n$$

が成り立つ。更に，$L^2(W)$ から $L^2(W)$ へのシフト作用素を U で表す。このとき，

$$U\left(\sum_{n=-\infty}^{\infty} a_n z^n\right)=\sum_{n=-\infty}^{\infty} a_n z^{n+1}$$

が成り立つ。このとき，$S=U|H^2(W)$ と定める。

$$S\left(\sum_{n=0}^{\infty} a_n z^n\right)=\sum_{n=0}^{\infty} a_n z^{n+1}$$

が成り立つ。

特に，W が恒等的に 1 に等しい定数関数のとき，$H^2(W)=H^2, L^2(W)=L^2$ が成り立つ。

Beurlingの定理　次の等式がなりたつ。

$$\{E\subset H^2 : E \text{ is not } S^*-\text{invariant but } S-\text{invariant}\}=\{qH^2 : q \text{ is an inner function}\}.$$

Wienerの定理　次の等式がなりたつ。

$$\{\mathrm{E}\subset L^2 : e^{i\theta}\mathrm{E}=\mathrm{E}\}=\{\chi_A L^2 : \mathrm{A} \text{ is a measurable set of } \mathbb{T}\}.$$

Sarasonの定理　内部関数 q に対して $N=H^2\ominus qH^2$ と定める。このとき，次の等式がなりたつ。

$$\{A\in B(N) : SA=AS\}=\left\{T_f : f\in H^\infty, \|A\|=\|f\|_\infty\right\}.$$

（cf. [2], [5], [7], [8], [14], [27], [35], [36], [41], [42], [44], [50], [51], [52], [56], [61]）

2．マジョライゼーションと因数分解定理

In this section, we use the Helson-Szegő type set $(HS)(r)$ to establish the condition of a, b, c and d satisfying $\|Af\|_2\geq\|Bf\|_2$ for all f in $\mathcal{P}$. The main theorem is Theorem 2.4. If $a, b, c, d\in L^\infty$, then this is equivalent to that B is majorized by A on L^2, i.e., $A^*A\geq B^*B$ on L^2, i.e., $\|Af\|_2\geq\|Bf\|_2$ for all f in L^2.

It is well known that the Helson-Szegő set

$$(HS)=\left\{e^{u+\tilde{v}}; u, v\in L^\infty \text{ are real functions, and } \|v\|_\infty<\frac{\pi}{2}\right\}$$

is equal to the set of all Muckenhoupt (A_2)-weights (cf. [5, p.254]). For a function r satisfying $0\leq r\leq 1$, we define the Helson-Szegő type set $(HS)(r)$ as follows.

$$(HS)(r)=\{\gamma e^{u+\tilde{v}}; \gamma \text{ is a positive constant}, u, v \text{ are real functions},$$
$$u\in L^1, v\in L^\infty, |v|\leq\pi/2, r^2e^u+e^{-u}\leq 2\cos v\}$$

Remark 2.1. If $|v|\leq\pi/2$, then $e^{\tilde{v}}\cos v\in L^1$ (cf. [5, p. 161]), and hence $e^{-\tilde{v}}\cos v\in L^1$. Therefore $(HS)(r)\subset\{W: W>0, r^2W\in L^1, W^{-1}\in L^1\}$. If $r^{-1}\in L^\infty$, then $(HS)(r)\subset(HS)$. In [10], we defined the another Helson-Szegő type set which is similar to $HS(r)$. If $r>0$, then W is in $(HS)(r)$ if and only if $W=\gamma e^{u+\tilde{v}}$ where γ is a positive constant, u, v are real functions such that there is a function t satisfying $1\leq t\leq r^{-1}$, $|v|\leq\cos^{-1}(rt)$ and $|u+\log r|\leq\cosh^{-1}(t)$.

Theorem 2.2. $|\phi|\geq W$ かつ $\int_{\mathbb{T}}(|\phi|-W)dm>0$ とし

$$r=|\phi|^{-1}\sqrt{|\phi|^2-W^2}$$

と定める。このとき，以下の条件は互いに同値である。

(1) There is a k in H^1 such that $|\phi-k|\leq W$.

(2) There is a non-zero k in H^1 such that $|\phi-k|\leq W$.

(3) $\log|\phi|\in L^1$ and there are real functions $u\in L^1$ and $v\in L^\infty$, $|v|\leq\pi/2$ such that $\phi e^{-u-\tilde{v}}$ is in $H^{1/2}$ and $r^2e^u+e^{-u}\leq 2\cos v$.

(4) $\log|\phi|\in L^1$ and there is a V in $(HS)(r)$ such that ϕ/V is in $H^{1/2}$.

Theorem 2.3. もし $|\phi|^2-W_1W_2\geq 0$ ならば，以下の条件は互いに同値である。

(1) For all $(Pf)\in\mathcal{P}_1$ and $(Qf)\in\mathcal{P}_2$,

$$\left|\int_{\mathbb{T}}(Pf)\overline{(Qf)}\phi dm\right|\leq\int_{\mathbb{T}}\frac{|Pf|^2W_1+|Qf|^2W_2}{2}dm,.$$

(2) W_1, W_2 and ϕ satisfy (2.1) or (2.2).

(2.1) $W_1\geq 0$, $W_2\geq 0$, and $|\phi|^2-W_1W_2=0$.

(2.2) $\log|\phi|\in L^1$, $W_1\geq 0$, $W_2\geq 0$, and there is a V in $(HS)(r)$ such that ϕ/V is in $H^{1/2}$, where $r=|\phi|^{-1}\sqrt{|\phi|^2-W_1W_2}$.

In Theorem 2.3, if $|\phi|^2-W_1W_2\leq 0$ then

$$\begin{aligned}\left|\int_{\mathbb{T}}(Pf)\overline{(Qf)}\phi dm\right| &\leq\int_{\mathbb{T}}|(Pf)(Qf)\phi|dm\\ &\leq\int_{\mathbb{T}}|(Pf)(Qf)|\sqrt{W_1W_2}dm\\ &\leq\int_{\mathbb{T}}\frac{|Pf|^2W_1+|Qf|^2W_2}{2}dm,\end{aligned}$$

and so (1) holds without the condition (2).

Theorem 2.4. 以下の条件は互いに同値である。

(1) For all f in $\mathcal{P}$,

$$\int_{\mathbb{T}} |a(Pf)+b(Qf)|^2\, dm \geq \int_{\mathbb{T}} |c(Pf)+d(Qf)|^2\, dm.$$

(2) a, b, c, d satisfy (2.1) or (2.2).

(2.1) $|a| \geq |c|, |b| \geq |d|$, and $ad-bc=0$.

(2.2) $\log\left|a\bar{b}-c\bar{d}\right| \in L^1$, $|a| \geq |c|, |b| \geq |d|$, $\log\left|a\bar{b}-c\bar{d}\right| \in L^1$, and there is a V in $(HS)(r)$ such that $\left(a\bar{b}-c\bar{d}\right)/V$ is in $H^{1/2}$, where $r=|ad-bc|/\left|a\bar{b}-c\bar{d}\right|$.

Let W be a non-negative function in L^1 such that $\int_{\mathbb{T}} W dm > 0$. Let $L^2(W)$ be the weighted Lebesgue space with the norm

$$\|f\|_{2,W} = \left\{\int_{\mathbb{T}} |f|^2 W dm\right\}^{1/2}.$$

Corollary 2.5. もし $A=aP+bQ$ かつ $B=cP+dQ$ ならば，以下の条件は互いに同値である。

(1) B is majorized by A on $L^2(W)$, i.e., $A^*A \geq B^*B$ on $L^2(W)$, i.e., $\|Af\|_{2,W} \geq \|Bf\|_{2,W}$ for all f in $L^2(W)$.

(2) There is a contraction C on $L^2(W)$ such that B is factorized as $B=CA$.

(3) a, b, c, d satisfy (3.1) or (3.2).

(3.1) $(|a|-|c|)W \geq 0$, $(|b|-|d|)W \geq 0$, and $(ad-bc)W=0$.

(3.2) $\log\left|a\bar{b}-c\bar{d}\right| W \in L^1$, $|a| \geq |c|, |b| \geq |d|$, and there is a V in $(HS)(r)$ such that $\left(a\bar{b}-c\bar{d}\right)W/V$ is in $H^{1/2}$, where $r=|ad-bc|/\left|a\bar{b}-c\bar{d}\right|$.

3．縮小作用素と有界特異積分作用素

In this section, some applications of Theorem 2.4 are given for contractive and bounded singular integral operators on weighted L^2 spaces.

Corollary 3.1. 以下の条件は互いに同値である。

(1) $\alpha P+\beta Q$ is contractive on $L^2(W)$.

(2) α, β and W satisfy (2.1) or (2.2).

(2.1) $\max(|\alpha|, |\beta|)W \leq W$ and $|\alpha-\beta|W=0$.

(2.2) $\log\left|1-\alpha\bar{\beta}\right| W \in L^1$, $\max(|\alpha|, |\beta|) \leq 1$, and there is a V in $(HS)(r)$ such that $\left(1-\alpha\bar{\beta}\right)W/V$ is in $H^{1/2}$, where $r=|\alpha-\beta|/\left|1-\alpha\bar{\beta}\right|$.

Corollary 3.2. （[10]）以下の条件は互いに同値である。

（1） $\alpha P+\beta Q$ is bounded on $L^2(W)$.

（2） α, β and W satisfy (2.1) or (2.2).

（2.1） $|\alpha-\beta|W=0$.

（2.2） There are real functions $u, v \in L^1$, and positive constant ε such that W is in $(HS)(r)$, where $r=\varepsilon|\alpha-\beta|$.

（3） α, β and W satisfy (3.1) or (3.2).

（3.1） $|\alpha-\beta|W=0$.

（3.2） There are real functions $u, v \in L^1$, and a positive constant γ such that $W=e^{u+\tilde{v}}$, $|v| \leq \pi/2$, $|\alpha-\beta|^2 e^u \leq \gamma \cos v$, and $e^{-u} \leq \gamma \cos v$.

Corollary 3.3. （[10]）以下の条件は互いに同値である。

（1） For all f in $\mathcal{P}$,

$$\int_{\mathbf{T}} |f|^2 W dm \geq \int_{\mathbf{T}} |Pf|^2 U dm.$$

（2） $\log W \in L^1$, $W \geq U$, and W is in $(HS)(r)$ where $r=\sqrt{U/W}$.

Corollary 3.4. （Koosis）以下の条件は互いに同値である。

（1） There is a non-negative function U in L^1 such that $\int_{\mathbf{T}} U dm > 0$ and for all f in $\mathcal{P}$,

$$\int_{\mathbf{T}} |f|^2 W dm \geq \int_{\mathbf{T}} |Pf|^2 U dm.$$

（2） There is a non-negative function U in L^1 such that $\log U \in L^1$ and for all f in $\mathcal{P}$,

$$\int_{\mathbf{T}} |f|^2 W dm \geq \int_{\mathbf{T}} |Pf|^2 U dm.$$

（3） W^{-1} is in L^1.

4． 拡大作用素と下に有界な特異積分作用素

In this section, some applications of Theorem 2.4 are given for expancive and bounded-below singular integral operators on weighted L^2 spaces.

Corollary 4.1. 以下の条件は互いに同値である。

（1） $\alpha P+\beta Q$ is expancive on $L^2(W)$.

（2） α, β and W satisfy (2.1) or (2.2).

（2.1）$\min(|\alpha|,|\beta|)W \geq W$ and $|\alpha-\beta|W=0$.

（2.2）$\log\left|\alpha\overline{\beta}-1\right|W \in L^1$, $\min(|\alpha|,|\beta|)\geq 1$, and there is a V in $(HS)(r)$ such that $\left(1-\alpha\overline{\beta}\right)W/V$ is in $H^{1/2}$, where $r=|\alpha-\beta|/\left|\alpha\overline{\beta}-1\right|$.

Corollary 4.2. 以下の条件は互いに同値である。

（1）$\alpha P+\beta Q$ is bounded-below on $L^2(W)$.

（2）α, β and W satisfy（2.1）or（2.2）.

（2.1）There is a positive constant ε such that $\min(|\alpha|,|\beta|)\mathrm{W} \geq \varepsilon W$, and $|\alpha-\beta|W=0$.

（2.2）There is a positive constant ε such that $\log\left|\alpha\overline{\beta}-\varepsilon\right|W \in L^1$, $\alpha^{-1}, \beta^{-1} \in L^\infty$, and there is a V in $(HS)(r)$ such that $\left(\alpha\overline{\beta}-\varepsilon\right)W/V$ is in $H^{1/2}$, where $r=|\alpha-\beta|\varepsilon/\left|\alpha\overline{\beta}-\varepsilon\right|$.

Corollary 4.3. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|\beta|dm>0$ とする。

（1）There is a constant $\varepsilon>0$ such that for all f in $\mathcal{P}$,

$$\int_{\mathbb{T}}|(\alpha P+\beta Q)f|^2 Wdm \geq \varepsilon\int_{\mathbb{T}}|Pf|^2 Wdm.$$

（2）$\log|\alpha\beta|W \in L^1$, $|a|\geq|c|$ and there is a V in $(HS)(r)$ such that $a\overline{b}/V$ is in $H^{1/2}$, where $r=|c/a|$.

Corollary 4.3 A. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|bc|dm>0$ とする。

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 dm \geq \int_{\mathbb{T}}|cPf|^2 dm.$$

（2）$\log|ab| \in L^1$, $|a|\geq|c|$ and there is a V in $(HS)(r)$ such that $a\overline{b}/V$ is in $H^{1/2}$, where $r=|c/a|$.

Corollary 4.4. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|\alpha-\beta|dm>0$ とする。

（1）$\alpha P+\beta Q$ is contractive on L^2.

（2）$\log\left|1-\alpha\overline{\beta}\right|dm \in L^1$, $\max(|\alpha|,|\beta|)\leq 1$, and there is a V in $(HS)(r)$ such that $\left(1-\alpha\overline{\beta}\right)/V$ is in $H^{1/2}$, where $r=|\alpha-\beta|/\left|1-\alpha\overline{\beta}\right|$.

Corollary 4.5. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|a-b|Wdm>0$ とする。

（1）For all f in $\mathcal{P}$,

$$\int_{\mathbb{T}}|f|^2 Wdm \geq \int_{\mathbb{T}}|(aP+bQ)f|^2 Udm.$$

（2）$\log\left|W-a\overline{b}U\right| \in L^1$, $\max\left(|a|^2,|b|^2\right)U \leq W$, and there is a V in $(HS)(r)$ such that $\left(W-a\overline{b}U\right)/V$ is in $H^{1/2}$, where $r=|a-b|\sqrt{WU}/\left|W-a\overline{b}U\right|$.

Theorem 4.6. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|bc|dm>0$ とする。

（1）For all f in $\mathcal{P}$,

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 Wdm \geq \int_{\mathbb{T}}|(cP+dQ)f|^2 Udm.$$

（2）$|a|^2 W \geq |c|^2 U$, $|b|^2 W \geq |d|^2 U$, and there is a positive function V in $(HS)(r)$ such that $\left(a\bar{b}W - c\bar{d}U\right)/V$ is a non-negative function in $H^{1/2}$, where $r^2 = |ad-bc|^2 WU / \left|a\bar{b}W - c\bar{d}U\right|^2$.

Corollary 4.7. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|\alpha|dm>0, |\alpha|\leq 1$　とする。

（1）For all f in $\mathcal{P}$,

$$\int_{\mathbb{T}}|\alpha Pf|^2 Wdm \leq \int_{\mathbb{T}}|f|^2 Wdm.$$

（2）There are real functions $u, v \in L^1$, and a positive constant γ such that $W = \gamma e^{u+\tilde{v}}, |v| \leq \pi/2$, and

$$|\alpha|^2 e^u + e^{-u} \leq 2\cos v.$$

Theorem 4.8. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|a-b|dm>0.$ とする。

（1）For all f in $\mathcal{P}$,

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 dm \leq \int_{\mathbb{T}}|f|^2 dm.$$

（2）$\max(|a|,|b|) \leq 1$, and there is a positive function V in $(HS)(r)$ such that $\left(1-a\bar{b}\right)/V$ is a non-negative function in $H^{1/2}$, where $r = |a-b|\sqrt{WU} / \left|W - a\bar{b}U\right|$.

Corollary 4.9. 以下の条件は互いに同値である。ただし，$\int_{\mathbb{T}}|ad-bc|dm>0, |a|\geq|c|, |b|\geq|d|,$ $a\bar{b}-c\bar{d}\geq 0$ とする。

（1）For all f in $\mathcal{P}$,

$$\int_{\mathbb{T}}|(aP+bQ)f|^2 dm \leq \int_{\mathbb{T}}|f|^2 dm.$$

（2）$\max(|a|, |b|) \leq 1$, and $1-a\bar{b}$ is in $(HS)(r)$, where $r^2 = |a-b|^2 / \left|1-a\bar{b}\right|^2$.

Theorem A. （Cotlar and Sadosky）以下の条件は互いに同値である。

（1）For all $(Pf) \in \mathcal{P}_1$ and $(Qf) \in \mathcal{P}_2$,

$$\int_{\mathbf{T}} \{|Pf|^2 W_1 + |Qf|^2 W_2 + 2Re\left((Pf)\overline{(Qf)}\phi\right)\} dm \geq 0.$$

（2）$W_1 \geq 0$, $W_2 \geq 0$, and there is an $k \in H^1$ such that $|\phi - k|^2 \leq W_1 W_2$.

Theorem 4.10. 以下の条件は互いに同値である。ただし，$\phi^2 - W_1W_2 \geq 0$ and $\int_{\mathbb{T}}(\phi^2 - W_1W_2)dm > 0$ とする。

（1） For all $(Pf) \in \mathcal{P}_1$ and $(Qf) \in \mathcal{P}_2$,

$$\int_{\mathbb{T}}\left\{|Pf|^2W_1 + |Qf|^2W_2 + 2Re\left((Pf)\overline{(Qf)}\phi\right)\right\}dm \geq 0.$$

（2） $W_1 \geq 0$, $W_2 \geq 0$, and there is a positive function V in $(HS)(r)$ and non-negative constant γ such that $\phi = \gamma V$, where $r = \left(|\phi|^2 - W_1W_2\right)/|\phi|^2$.

Let m be a normalized Lebesgue measure $\dfrac{d\theta}{2\pi}$ on the unit circle $\mathbb{T}$. Let $a, b \in L^\infty = L^\infty(m)$ satisfy $|a-b| > 0$. Let $w \in L^1 = L^1(m)$ satisfy $w > 0$. The Riesz projection P and $Q = I - P$ can be written as $P = \dfrac{I+S}{2}$, $Q = \dfrac{I-S}{2}$, where

$$(Sf)(\zeta) = \frac{1}{\pi i}\int_{\mathbb{T}}\frac{f(z)}{z-\zeta}dz,$$

in the sense of Cauchy's principal value integral, where $\zeta \in \mathbb{T}$, and $f \in L^1$. Hence $P+Q=I$ and $P-Q=S$. Let $\tilde{f}$ denote the harmonic conjugate function of f. Then $\tilde{f} = -i\left(Sf - \hat{f}(0)\right)$, where $\hat{f}(0)$ is the 0-th Fourier coefficient of f.

For $1 < p < \infty$, $L^p(w)$-norm is:

$$\|f\|_{p,w} = \|f\|_{L^p(w)} = \left(\int_{\mathbb{T}}|f|^p w\,dm\right)^{1/p}$$

If $p=2$, then we write $\|f\|_w = \|f\|_{2,w}$.

$\boxed{\text{Helson-Szegő theorem}}$ （1960）

$\|Pf\|_w \leq c\|f\|_w$, (for all f), for some positive constant c.

$\Leftrightarrow dist\left(\dfrac{\overline{h}}{h}, H^\infty\right) < 1$, where h is an outer function satisfying $w = |h|^2$ a.e.

$\Leftrightarrow$ There are real functions $u, v \in L^\infty$, $|v| < \pi/2$ such that $w = \exp(u + \tilde{v})$ a.e.

$\boxed{\text{Widom-Devinatz theorem}}$ （1960）

Let $|a| = 1$ a.e. Then

（1） $\|(aP+Q)f\|_w \geq \varepsilon\|f\|_w$, (for all f), for some positive constant ε.

$$\Leftrightarrow dist(a, H^{\infty})<1.$$

(2) $aP+Q$ has a bounded inverse operator on L^2.

$\Leftrightarrow$ There is an outer function k such that $\|a-k\|_{\infty}<1$.

$\Leftrightarrow$ There are real functions $u, v\in L^{\infty}, |v|<\pi/2$ and a complex constant $\gamma, |\gamma|=1$ such that $a=\gamma\exp(i(\tilde{u}+v))$ a.e.

$\boxed{\text{Hunt-Muckenhoupt-Wheeden theorem}}$ (1973)

Let $1<p<\infty$. Then

$\|Pf\|_{p,w}\leq c\|f\|_{p,w}$,(for all f), for some positive constant c.

$$\Leftrightarrow sup_I\left(\frac{1}{m(I)}\int_I wdm\right)\left(\frac{1}{m(I)}\int_I w^{-1/(p-1)}dm\right)^{p-1}<\infty$$

$\boxed{\text{Rochberg theorem}}$ (1977)

Let $1<p<\infty, a\in L^{\infty}, w\in(A_p)$. Then

$aP+Q$ has a bounded inverse operator on $L^p(w)$.

$\Leftrightarrow$ There are real functions $U\in L^{\infty}, V\in L^1$ and a complex constant $\gamma, |\gamma|=1$ such that $w\exp\left(\frac{p}{2}V\right)\in(A_p)$, $a=\gamma\exp\left(U-i\tilde{V}\right)$ a.e.

$\boxed{\text{Cotlar-Sadosky Theorem}}$ (1979)

Let c be a constant satisfying $c\geq 1$. Then

$\|Sf\|_w\leq c\|f\|_w$,(for all f).

$\Leftrightarrow$ There is a $k\in H^1$ such that $|w-k|^2\leq\left(1-\left(\frac{2c}{c^2+1}\right)^2\right)w^2$ a.e.

$\Leftrightarrow$ There are real functions $u,v\in L^{\infty}$ such that $w=\exp(u+\tilde{v}+const.)$,

$|v|\leq\frac{\pi}{2}-\arcsin\left(\frac{2c}{c^2+1}\right)$, $|u|\leq\cosh^{-1}\left(\frac{c^2+1}{2c}\cos v\right)$ a.e. where

$$\cosh^{-1}x=\log\left(x+\sqrt{x^2-1}\right).$$

$\boxed{\text{Nakazi theorem}}$ (1993)

$1<p<\infty, a\in L^{\infty}, w\in(A_p)$　とする。

(1) Let h be an outer function satisfying $w=|h|^p$ a.e. Then
$\|(aP+Q)f\|_{p,w}\geq\varepsilon\|f\|_{p,w}$,(for all f), for some positive constant ε.
$\Leftrightarrow$ There are functions k and a_0 such that $a=k(h/\bar{h})a_0$ a.e. where $k,k^{-1}\in H^\infty$, $|a_0|=1$ a.e., and
$\|(a_0P+Q)f\|_p\geq\varepsilon\|f\|_p$,(for all f), for some positive constant ε.

(2) There is an inner function q such that $\bar{q}aP+Q$ has a bounded inverse operator on $L^p(w)$.
$\Rightarrow\|(aP+Q)f\|_{p,w}\geq\varepsilon\|f\|_{p,w}$, (for all f), for some positive constant ε.
If $p=2$, then the converse is also true.

Let $a,b\in L^\infty$, and let c be a positive constant. Then we consider the positive function w in L^1 satisfying $\|(aP+bQ)f\|_w\leq c\|f\|_w$,(for all f).

Let $|a\bar{b}-c^2|>0$ a.e, and let

$$d_c=d_c(a,b)=\left|\frac{(a-b)c}{a\bar{b}-c^2}\right|.$$

Since
$\|aP+bQ\|_{B(L^2(w))}\leq c\Rightarrow\max(|a|,|b|)\leq c$ a.e.,
we have $|d_c(a,b)|\leq 1$ a.e. If $\max(|a|,|b|)\leq c$ a.e., then
$\|aP+bQ\|_{B(L^2(w))}\leq c\Leftrightarrow$ There is a $k\in H^1$ such that $\left|1-\frac{k}{(a\bar{b}-c^2)w}\right|^2\leq 1-d_c^2$ a.e.
If a,b are complex constants, then

$$c=\|aP+bQ\|_{B(L^2(w))}\Rightarrow d_c(a,b)=\frac{1}{\|P\|_{B(L^2(w))}}$$

Hence we have the Cotlar-Sadosky's result:

$$c=\|S\|_{B(L^2(w))}\Rightarrow c=\|P-Q\|_{B(L^2(w))}\Rightarrow d_c(1,-1)=\frac{2c}{c^2+1}=\frac{1}{\|P\|_{B(L^2(w))}}.$$

Theorem 4.11. 以下の条件は互いに同値である。
(1) $\|(aP+bQ)f\|_w\leq c\|f\|_w$, (for all f).
(2) There is an inner function q and real functions $t,u\in L^1$, $v\in L^\infty$ such that

$$w=\frac{1}{|a\bar{b}-c^2|}\exp(u+\tilde{v}+t)\text{ a.e., }\frac{a\bar{b}-c^2}{|a\bar{b}-c^2|}=q\exp(i\tilde{t})\text{ a.e.,}$$

$$|v|\leq\frac{\pi}{2}-\arcsin d_c\text{ a.e., }d_c^2e^u+e^{-u}\leq 2\cos v\text{ a.e.}$$

（3） There is an inner function q and real functions $t \in L^1, v \in L^\infty, u_0$ such that

$$w = \frac{1}{|a-b|}\exp(u_0 + \tilde{v} + t) \text{ a.e.,} \quad \frac{a\bar{b}-c^2}{|a\bar{b}-c^2|} = q\exp(i\tilde{t}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - arcsin\, d_c \text{ a.e.,} \quad |\mathrm{u}_0| \leq \cosh^{-1}\left(\frac{\cos \mathrm{v}}{\mathrm{d_c}}\right) \text{ a.e.}$$

Theorem 4.12. 以下の条件は互いに同値である。ただし $|a-b|>0$ a.e., $\max(|a|,|b|) \leq c$ a.e., $|a\bar{b}-c^2|>0$ a.e. とする。

$$\exp(-s)|a\bar{b}-c^2|w \in L^1, \quad \gamma\exp(i\tilde{s}) = \frac{a\bar{b}-c^2}{|a\bar{b}-c^2|} \text{ a.e.}$$

Then （1）～（3） are equivalent.

（1） $\|(aP+bQ)f\|_w \leq c\|f\|_w$, (for all f).

（2） There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \frac{1}{|a\bar{b}-c^2|}\exp(u + \tilde{v} + s + const.) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.,} \quad d_c^2 e^u + e^{-u} \leq 2\cos v \text{ a.e.}$$

（3） There are real functions $v \in L^\infty, u$ such that

$$w = \frac{1}{|a-b|}\exp(u_0 + \tilde{v} + s + const.) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.,} \quad |u_0| \leq \cosh^{-1}\left(\frac{\cos v}{d_c}\right) \text{ a.e.}$$

Theorem 4.13. 以下の条件は互いに同値である。ただし $|a-b|>0$ a.e., $\max(|a|,|b|) \leq c$ a.e., $a\bar{b} \in H^\infty, a\bar{b} \neq c^2$ とする。

（1） $\|(aP+bQ)f\|_w \leq c\|f\|_w$, (for all f).

（2） There are real functions $u \in L^1, v \in L^\infty$ such that

$$w = \exp(u + \tilde{v} + const.) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2} - \arcsin d_c \text{ a.e.,} \quad d_c^2 e^u + e^{-u} \leq 2\cos v \text{ a.e.}$$

（3） There are real functions $v \in L^\infty, u_0$ such that

$$w = d_c^{-1}\exp(u_0 + \tilde{v} + const.) \text{ a.e.,}$$

$$|v| \le \frac{\pi}{2} - \arcsin d_c \text{ a.e.,} \quad |u_0| \le \cosh^{-1}\left(\frac{\cos v}{d_c}\right) \text{ a.e.}$$

By Theorem 4.14, we generalize the next result. Let $c>1$, and let $w(z)=\frac{1}{|z-c^2|^2}$. Then $(c^2-\bar{z})w=\frac{1}{c^2-z}\in H^1$. Hence,

$$\|(\bar{z}P+Q)f\|_w^2 = c^2\|f\|_w^2 - (c^2-1)\left(\|Pf\|_w^2 + \|Qf\|_w^2\right)$$
$$\le c^2\|f\|_w^2, \text{(for all } f).$$

Theorem 4.14. 以下の条件は互いに同値である。ただし $|a-b|>0$ a.e., $\max(|a|,|b|)\le c$ a.e., $\bar{a}b\in H^\infty$, $\bar{a}b\ne c^2$. とする。

(1) $\|(aP+bQ)f\|_w \le c\|f\|_w$, (for all f).

(2) There are realfunctions $u\in L^1$, $v\in L^\infty$ such that

$$w=\frac{1}{|a\bar{b}-c^2|^2}\exp(u+\tilde{v}+const.) \text{ a.e.,}$$

$$|v| \le \frac{\pi}{2} - \arcsin d_c \text{ a.e.,} \quad d_c^2 e^u + e^{-u} \le 2\cos v \text{ a.e.}$$

(3) There are real functions $v\in L^\infty$, u_0 such that

$$w=\frac{1}{|a-b|\cdot|a\bar{b}-c^2|}\exp(u_0+\tilde{v}+const.) \text{ a.e.,}$$

$$|v| \le \frac{\pi}{2} - \arcsin d_c \text{ a.e.,} \quad |u_0| \le \cosh^{-1}\left(\frac{\cos v}{d_c}\right) \text{ a.e.}$$

Let ε be a positive constant, and let $a, b\in L^\infty$. Then we consider the positive function w in L^1 satisfying

$$\|(aP+bQ)f\|_w \ge \varepsilon\|f\|_w, \text{ (for all } f).$$

Let $|a\bar{b}-\varepsilon^2|>0$ a.e., and let

$$d_\varepsilon(a,b)=\left|\frac{(a-b)\varepsilon}{a\bar{b}-\varepsilon^2}\right|.$$

Since
$\|(aP+bQ)f\|_w \ge \varepsilon\|f\|_w$,(for all f)$\Rightarrow \min(|a|,|b|)\ge\varepsilon$ a.e.,
it follows that $|d_\varepsilon(a,b)|\le 1$ a.e. Hence, if $\min(|a|,|b|)\ge\varepsilon$ a.e., then

$$\|(aP+bQ)f\|_w \geq \varepsilon\|f\|_w, (\text{for all } f)$$

$\Leftrightarrow$ There is a $k \in H^1$ such that

$$\left|1-\frac{k}{(a\bar{b}-\varepsilon^2)w}\right|^2 \leq 1-d_\varepsilon^2 \text{ a.e.} \quad \text{(cf.[8], [9], [13]).}$$

Theorem 4.15. 以下の条件は互いに同値である。ただし $|a-b|>0$ a.e., $\min(|a|,|b|)\geq\varepsilon$ a.e., $|a\bar{b}-\varepsilon^2|>0$ a.e. とする。

(1) $\|(aP+bQ)f\|_w \geq \varepsilon\|f\|_w, (\text{for all } f)$.

(2) There is an inner function q and real functions $t, u \in L^1, v \in L^\infty$ such that

$$w=\frac{1}{|a\bar{b}-\varepsilon^2|}\exp(u+\tilde{v}+t) \text{ a.e.,} \quad \frac{a\bar{b}-\varepsilon^2}{|a\bar{b}-\varepsilon^2|}=q\exp(i\tilde{t}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2}-\arcsin d_\varepsilon \text{ a.e.,} \quad d_\varepsilon^2 e^u + e^{-u} \leq 2\cos v \text{ a.e.}$$

(3) There is an inner function q and real functions $t \in L^1, v \in L^\infty, u_0$ such that

$$w=\frac{1}{|a-b|}\exp(u_0+\tilde{v}+t) \text{ a.e.,} \quad \frac{a\bar{b}-\varepsilon^2}{|a\bar{b}-\varepsilon^2|}=q\exp(i\tilde{t}) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2}-\arcsin d_\varepsilon \text{ a.e.,} \quad |u_0| \leq \cosh^{-1}\left(\frac{\cos v}{d_\varepsilon}\right) \text{ a.e.}$$

Theorem 4.16 以下の条件は互いに同値である。ただし $|a-b|>0$ a.e., $\min(|a|,|b|)\geq\varepsilon$ a.e., $|a\bar{b}-\varepsilon^2|>0$ a.e. とし $s \in L^1$ と complex constant $\gamma, |\gamma|=1$ が存在して

$$\exp(-s)|a\bar{b}-\varepsilon^2|w \in L^1, \quad \gamma\exp(i\tilde{s})=\frac{a\bar{b}-\varepsilon^2}{|a\bar{b}-\varepsilon^2|} \text{ a.e.}$$

を満足していることを仮定する。

(1) $\|(aP+bQ)f\|_w \geq \varepsilon\|f\|_w, (\text{for all } f)$.

(2) There are real functions $u \in L^1, v \in L^\infty$ such that

$$w=\frac{1}{|a\bar{b}-\varepsilon^2|}\exp(u+\tilde{v}+s+const.) \text{ a.e.,}$$

$$|v| \leq \frac{\pi}{2}-\arcsin d_\varepsilon \text{ a.e.,} \quad d_\varepsilon^2 e^u + e^{-u} \leq 2\cos v \text{ a.e.}$$

(3) There are real functions $v \in L^\infty, u$ such that

$$w=\frac{1}{|a-b|}\exp(u_0+\tilde{v}+s+const.) \text{ a.e.,}$$

$$|v| \le \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e.}, \quad |u| \le \cosh^{-1}\left(\frac{\cos v}{d_\varepsilon}\right) \text{ a.e.}$$

By Theorem 4.17, we generalize the next result. Let $0<\varepsilon\le 1$, and let

$$w(z) = \left|\frac{z-1}{z-\varepsilon^2}\right|^2.$$

Since

$$(z-\varepsilon^2)w = \frac{(z-1)(\bar{z}-1)}{\bar{z}-\varepsilon^2} = \frac{-(1-z)^2}{1-\varepsilon^2 z} \in H^1,$$

it follows that

$$\|(zP+Q)f\|_w^2 = \varepsilon^2\|f\|_w^2 + (1-\varepsilon^2)\left(\|Pf\|_w^2 + \|Qf\|_w^2\right) \ge \varepsilon^2\|f\|_w^2, \text{(for all } f).$$

Theorem 4.17. $0<\varepsilon<1$, $a, c\in L^\infty$, $!|a-b|>0$ a.e., $\min(|a|, |b|)\ge\varepsilon$ a.e. とし $a\bar{b}$ は non-constant inner function satisfying

$$\left|\frac{a\bar{b}-\varepsilon^2}{a\bar{b}-1}\right|^2 w \in L^1$$

とする。このとき $aP+bQ$ は unbounded operator であり，以下の条件は互いに同値である。

(1) $\|(aP+bQ)f\|_w \ge \varepsilon\|f\|_w$, (for all f).

(2) There are real functions $u\in L^1, v\in L^\infty$ such that

$$w = \left|\frac{a\bar{b}-1}{a\bar{b}-\varepsilon^2}\right|^2 \exp\left(u+\tilde{v}+const.\right) \text{ a.e.},$$

$$|v| \le \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e.}, \quad d_\varepsilon^2 e^u + e^{-u} \le 2\cos v \text{ a.e.}$$

(3) There are real functions $v\in L^\infty, u_0$ such that

$$w = \frac{\left|a\bar{b}-1\right|^2}{|a-b|\cdot\left|a\bar{b}-\varepsilon^2\right|} \exp\left(u_0+\tilde{v}+const.\right) \text{ a.e.},$$

$$|v| \le \frac{\pi}{2} - \arcsin d_\varepsilon \text{ a.e.}, \quad |u_0| \le \cosh^{-1}\left(\frac{\cos v}{d_\varepsilon}\right) \text{ a.e.}$$

Theorem 4.18. もし $|a-b|>0$ a.e. であり，$\bar{a}b$ が non-constant inner function であるならば $aP+bQ$ は not bounded-below in $L^2(w)$ である。

Theorem 4.17 has the following equivalent formulation. For $a, b, c, d\in L^\infty$, the condition (1) of Theorem 4.18 implies that B is majorized by A on L^2, i.e., $A^*A \ge B^*B$ on $L^2(W)$, where

$A=aP+bQ$ and $B=cP+dQ$.

Corollary 4.19. $W\in(HS)$ とし α,β は complex functions in L^∞ であり $|\alpha|=|\beta|$ を満足しているとする。このとき，以下の条件は互いに同値である。

（1）$\alpha P+\beta Q$ is bounded-below on $L^2(W)$.

（2）$\alpha^{-1}\in L^\infty$, and there is a V in (HS) such that $\alpha W/V$ is in $H^{1/2}$.

Corollary 4.20. 以下の条件は互いに同値である。

（1）There is a non-negative function U in L^1 such that for all f in $\mathcal{P}$,

$$\int_{\mathbf{T}}|f|^2 Udm\geq\int_{\mathbf{T}}|Pf|^2 Wdm.$$

（2）There is a positive constant γ and real function v sastisfying $|v|<\pi/2$ and $W\leq\gamma e^{\tilde{v}}\cos v$.

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