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On Weight Functions and Norms of Some Singular Integral Operators

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Dedicated to Professor Takahiko Nakazi on the occasion of his 60th birthday

ABSTRACT

Let m be a normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let a and b be bounded m -measurable functions on $\mathbf{T}$, and let w be a positive and integrable function on $\mathbf{T}$. It is well known that the Riesz projection P_+ is bounded on the weighted space $L^2(wdm)$ if and only if w has the Helson-Szegő representation. Let $P_- = I - P_+$, where I denotes the identity operator. In this paper, if the singular integral operator $aP_+ + bP_-$ is bounded and invertible on the weighted space $L^2(wdm)$, then we establish the Helson-Szegő type representation of the weight function w using the operator norms of $aP_+ + bP_-$ and $(aP_+ + bP_-)^{-1}$.

KEYWORDS: Singular integral operator, Riesz projection, Norm, Hardy space, Weight function, Helson-Szegő weight.

MSC (2000): Primary 45E10, 47B35; Secondary 46J15.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $\mathbf{T} = \{z; |z|=1\}$. That is, $dm(\zeta) = \frac{d\theta}{2\pi}$ for $\zeta = e^{i\theta}$. Throughout this paper, we always assume that the weight function w satisfies $w > 0$ a.e. on $\mathbf{T}$, and $w \in L^1(\mathbf{T}) = L^1(\mathbf{T}, dm)$. Let $\mathcal{P}$ denote the set of all trigonometric polynomials. Define the Riesz projection P_+ by

$$(P_+f)(e^{it}) = \sum_{k \geq 0} \hat{f}(k) e^{ikt}, \quad f \in \mathcal{P},$$

where $\hat{f}(k)$ denotes the k -th Fourier coefficient of f . Let S be the singular integral operator (SIO) defined by

$$(Sf)(\zeta) = \frac{1}{\pi i} \int_{\mathbf{T}} \frac{f(z)}{z - \zeta} dz,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [13], p.11). If f is in L^1 , then $(Sf)(\zeta)$ exists for almost everywhere ζ on $\mathbf{T}$, and Sf becomes a measurable function on $\mathbf{T}$. Let $P_- = I - P_+$, where I denotes the identity operator. Then $P_+^2 = P_+$, $P_-^2 = P_-$, $P_+ = \frac{I+S}{2}$, and $P_- = \frac{I-S}{2}$. Since $S = P_+ - P_-$, $S^2 = I$. The harmonic conjugate function $\tilde{f}$ of f is defined by

$$\tilde{f}(e^{i\theta}) = \int_{-\pi}^{\pi} \cot\left(\frac{\theta-t}{2}\right) f(e^{it}) \frac{dt}{2\pi},$$

where the integral is understood in the sense of Cauchy's principal value. The Hilbert transform H is defined by $H = -i(P_+ - P_-)$. Hence $H = -iS$. Then

$$\begin{aligned} \tilde{f} &= -i(P_+f - P_-f) + i\hat{f}(0) \\ &= -i(Sf - \hat{f}(0)) \\ &= Hf + i\hat{f}(0). \end{aligned}$$

In particular if $\hat{f}(0)=0$, then $\tilde{f}=Hf$. The weighted L^2 -norm is defined by

$$\|f\|_w = \|f\|_{L^2(wdm)} = \left(\int_{\mathbf{T}} |f|^2 w dm \right)^{1/2}.$$

Let $a, b \in L^\infty(\mathbf{T}) = L^\infty(\mathbf{T}, dm)$. If $aP_+ + bP_-$ is a bounded operator on $L^2(wdm)$, then the operator norm of $aP_+ + bP_-$ is defined by

$$\|aP_+ + bP_-\| = \|aP_+ + bP_-\|_{B(L^2(w))} = \sup\{\|(aP_+ + bP_-)f\|_w; f \in L^2(w), \|f\|_w = 1\}.$$

H. Helson and G. Szegő proved that the Riesz projection P_+ is bounded on $L^2(d\mu)$ if and only if $d\mu = wdm$ is an (HS) measure (cf. [3], [12], [14], [15], [20], [21], [26]). Let H^1 denote the Hardy space. H^1 is a norm-closed subspace of $L^1(\mathbf{T})$. There is a deep extension of the Helson-Szegő approach developed in a series of papers by M. Cotlar, C. Sadosky, R. Arocena and M. Dominguez (cf. [21] Vol.1, p.132, [24], [4], [2], [1], [23], [5], [8], [6], [25], [7]). P. Koosis [16] established the two weights norm inequality for the Hilbert transform on weighted L^2 spaces. In particular, M. Cotlar and C. Sadosky proved the following:

Theorem A. (Cotlar-Sadosky) *For a positive constant M satisfying $M \geq 1$, the following are equivalent:*

- (i) *the Hilbert transform H is bounded on $L^2(wdm)$ and $\|H\| \leq M$;*
- (ii) *there exists $h \in H^1$ such that $|w - h| \leq \frac{M^2 - 1}{M^2 + 1} w = \left(1 - \frac{2}{M^2 + 1}\right) w$ a.e. on $\mathbf{T}$;*

(iii) there exists $h \in H^1$ such that $|w - h|^2 \leq \left(1 - \left(\frac{2M}{M^2+1}\right)^2\right)w^2$ a.e. on $\mathbf{T}$;

(iv) there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$,

$$\|v\|_\infty \leq \arccos\left(\frac{2M}{M^2+1}\right) = \frac{\pi}{2} - \arcsin\left(\frac{2M}{M^2+1}\right) \text{ and } |u| \leq \cosh^{-1}\left(\frac{M^2+1}{2M}\cos v\right) \text{ a.e. on } \mathbf{T},$$

where $\cosh^{-1}x = \log(x + \sqrt{x^2 - 1})$.

If the SIO $(aP_+ + bP_-)$ is bounded and invertible on the weighted space $L^2(wdm)$, then we shall consider the explicit form of the (HS) type weights w using functions $\arccos$, $\arcsin$ and $\cosh^{-1}$ (cf. [19], [28], [29], [30]). In this paper, Theorems 3 and 4 are the main theorems.

2. MAIN THEOREMS AND COROLLARIES

In this section, we shall give the main theorems and their corollaries. We use Lemmas 1 and 2 to prove the main theorems. The proof of Lemma 1 is essential (cf. [19], [28], [29], [30]), which uses the Cotlar-Sadosky lifting theorem. There are many kinds of proofs of the lifting theorem.

Lemma 1. For $a, b \in L^\infty$ and a positive constant M satisfying $\max(|a|, |b|) \leq M$, let $d = \left|\frac{(a-b)M}{M^2 - a\bar{b}}\right|$, then $\|d\|_\infty \leq 1$, and the following are equivalent:

- (i) the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ and $\|aP_+ + bP_-\| \leq M$;
- (ii) there exists $h \in H^1$ such that $|(M^2 - a\bar{b})w - h|^2 \leq (M^2 - |a|^2)(M^2 - |b|^2)w^2$ a.e. on $\mathbf{T}$;
- (iii) there exists $h \in H^1$ such that $|w - h|^2 \leq (1 - d^2)w^2$ a.e. on $\mathbf{T}$;
- (iv) there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e.,
 $|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d$ a.e. and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

The proof of Lemma 1 is similar to the proof of Theorem A, and the proof of the following Lemma 2 is similar to one of Lemma 1.

Lemma 2. For $a, b \in L^\infty$ and a positive constant N satisfying $\min(|a|, |b|) \geq N$, let $d = \left|\frac{(a-b)N}{N^2 - a\bar{b}}\right|$, then $\|d\|_\infty \leq 1$, and the following are equivalent:

- (i) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w$ for every $f \in \mathcal{P}$;
- (ii) there exists $h \in H^1$ such that $|(N^2 - a\bar{b})w - h|^2 \leq (N^2 - |a|^2)(N^2 - |b|^2)w^2$ a.e. on $\mathbf{T}$;

- (iii) there exists $h \in H^1$ such that $|w - h|^2 \leq (1 - d^2)w^2$ a.e. on $\mathbf{T}$;
- (iv) there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e.,
 $|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d$ a.e. and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

The following Theorems 3 and 4 are the main theorems of this paper. These follow from Lemmas 1 and 2.

Theorem 3. For $a, b \in L^\infty$ and positive constants M and N satisfying $|M^2 - a\bar{b}| \cdot |N^2 - a\bar{b}| > 0$ and $N \leq \min(|a|, |b|) \leq \max(|a|, |b|) \leq M$, let $d = \max\left(\left|\frac{(a-b)M}{M^2 - a\bar{b}}\right|, \left|\frac{(a-b)N}{N^2 - a\bar{b}}\right|\right)$.

Then $\|d\|_\infty \leq 1$, and the following are equivalent:

- (i) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w \leq M\|f\|_w$ for every $f \in \mathcal{P}$;
- (ii) there exists $h \in H^1$ such that $|(M^2 - a\bar{b})w - h|^2 \leq (M^2 - |a|^2)(M^2 - |b|^2)w^2$ a.e. on $\mathbf{T}$;
- (iii) there exists $h \in H^1$ such that $|w - h|^2 \leq (1 - d^2)w^2$ a.e. on $\mathbf{T}$;
- (iv) there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e.,
 $|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d$ a.e. and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

Theorem 4 follows from Theorem 3. If $w dm$ is the (HS) measure, then there are many articles concerning the invertibility of the SIO $(aP_+ + bP_-)$ on $L^2(w dm)$ (cf. [3], [9], [10], [13], [17], [21], [22]). In Theorems 3 and 4, we do not assume that $w dm$ is the (HS) measure. In Theorem 4, we establish the (HS) type representation of the weight function w using the operator norms of $aP_+ + bP_-$ and $(aP_+ + bP_-)^{-1}$.

Theorem 4. For $a, b \in L^\infty$ such that the SIO $(aP_+ + bP_-)$ is bounded and invertible on $L^2(w dm)$ and $\| \|aP_+ + bP_- \|^2 - a\bar{b} \| \cdot \left| \frac{1}{\|(aP_+ + bP_-)^{-1}\|^2} - a\bar{b} \right| > 0$ a.e. on $\mathbf{T}$, let

$$d = \max\left(\left|\frac{(a-b)\|aP_+ + bP_-\|}{\|aP_+ + bP_-\|^2 - a\bar{b}}\right|, \left|\frac{(a-b)\|(aP_+ + bP_-)^{-1}\|}{1 - a\bar{b}\|(aP_+ + bP_-)^{-1}\|^2}\right|\right).$$

Then $\|d\|_\infty \leq 1$, and there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e.,
 $|v| \leq \arccos d = \frac{\pi}{2} - \arcsin d$ a.e. and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

The following Corollaries 5, 6 and 7 follow from Theorem 3.

Corollary 5. For distinct and non-zero complex constants a, b and positive constants M, N satisfying $|M^2 - a\bar{b}| \cdot |N^2 - a\bar{b}| > 0$ and $N \leq \min(|a|, |b|) \leq \max(|a|, |b|) \leq M$, let

$d = \max\left(\left|\frac{(a-b)M}{M^2 - a\bar{b}}\right|, \left|\frac{(a-b)N}{N^2 - a\bar{b}}\right|\right)$, then $d \leq 1$, and the following are equivalent:

- (i) $N\|f\|_w \leq \|(aP_+ + bP_-)f\|_w \leq M\|f\|_w$ for every $f \in \mathcal{P}$;
- (ii) there exist $h, k \in H^1$ such that $|(M^2 - a\bar{b})w - h|^2 \leq (M^2 - |a|^2)(M^2 - |b|^2)w^2$ a.e., and $|(N^2 - a\bar{b})w - k|^2 \leq (N^2 - |a|^2)(N^2 - |b|^2)w^2$ a.e. on $\mathbf{T}$;
- (iii) there exists $h \in H^1$ such that $|w - h|^2 \leq (1 - d^2)w^2$ a.e. on $\mathbf{T}$;
- (iv) there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e., $\|v\|_\infty \leq \arccos d = \frac{\pi}{2} - \arcsin d$ and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

Corollary 6. For distinct and non-zero complex constants a, b and a positive constant M satisfying $\max(|a|, |b|) \leq M$, let $d = \left|\frac{(a-b)M}{M^2 - a\bar{b}}\right|$, then $d \leq 1$, and the following are equivalent:

- (i) the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ and $\|aP_+ + bP_-\| \leq M$;
- (ii) the SIO $(aP_+ + bP_-)^{-1} = a^{-1}P_+ + b^{-1}P_-$ is bounded on $L^2(wdm)$ and $\|(aP_+ + bP_-)^{-1}\| \leq \frac{M}{|ab|}$;
- (iii) there exists $h \in H^1$ such that $|(M^2 - a\bar{b})w - h|^2 \leq (M^2 - |a|^2)(M^2 - |b|^2)w^2$ a.e. on $\mathbf{T}$;
- (iv) there exists $h \in H^1$ such that $|w - h|^2 \leq (1 - d^2)w^2$ a.e. on $\mathbf{T}$;
- (v) there exist real functions $u, v \in L^\infty$ such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e., $\|v\|_\infty \leq \arccos d = \frac{\pi}{2} - \arcsin d$ and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

Corollary 7. For positive constants M and N satisfying $N \leq 1 \leq M$, let $\gamma = \min(M, N^{-1})$, $\delta = \max\left(\frac{2}{M^2 + 1}, \frac{2N^2}{N^2 + 1}\right)$ and $d = \max\left(\frac{2M}{M^2 + 1}, \frac{2N}{N^2 + 1}\right)$. Then $\delta = \frac{2}{\gamma^2 + 1}$, $d = \frac{2\gamma}{\gamma^2 + 1}$, and the following are equivalent:

- (i) $N\|f\|_w \leq \|Hf\|_w \leq M\|f\|_w$ for every $f \in \mathcal{P}$;
- (ii) there exists $h \in H^1$ such that $|w - h| \leq (1 - \delta)w$ a.e. on $\mathbf{T}$;
- (iii) there exists $h \in H^1$ such that $|w - h|^2 \leq (1 - d^2)w^2$ a.e. on $\mathbf{T}$;
- (iv) there exist real functions $u, v \in L^\infty$ such that $w = \exp(u + \tilde{v} + \text{const.})$, $\|v\|_\infty \leq \arccos\left(\frac{2\gamma}{\gamma^2 + 1}\right) = \frac{\pi}{2} - \arcsin\left(\frac{2\gamma}{\gamma^2 + 1}\right)$ and $|u| \leq \cosh^{-1}\left(\frac{\gamma^2 + 1}{2\gamma} \cos v\right)$ a.e. on $\mathbf{T}$.

For distinct and non-zero complex constants a and b , the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ if and only if $(aP_+ + bP_-)$ is invertible on $L^2(wdm)$ if and only if the Riesz projection P_+ is bounded on $L^2(wdm)$. Then

$$\|(aP_+ + bP_-)^{-1}\| = \|a^{-1}P_+ + b^{-1}P_-\| = \frac{1}{|ab|} \|bP_+ + aP_-\| = \frac{1}{|ab|} \|aP_+ + bP_-\|.$$

By Theorem 4, we have:

Corollary 8. *For distinct and non-zero complex constants a and b , the SIO $(aP_+ + bP_-)$ is bounded on $L^2(wdm)$ if and only if $(aP_+ + bP_-)$ is invertible. Let*

$$d = \left| \frac{(a-b)\|aP_+ + bP_-\|}{\|aP_+ + bP_-\|^2 - ab} \right|.$$

Then $0 < d \leq 1$, and there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e., $\|v\|_\infty \leq \arccos d = \frac{\pi}{2} - \arcsin d$ and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right)$ a.e. on $\mathbf{T}$.

Example. The relation between the norms of the operators H, P_+, P_- on the space $L^2(w)$;

$$\|P_+\| = \|P_-\| = \frac{\|H\| + \|H\|^{-1}}{2}$$

was remarked by Spitkovsky [27]. Then

$$\|H\| = \|S\| = \|P_+ - P_-\| = \|P_+\| + \sqrt{\|P_+\|^2 - 1}.$$

For $\zeta_0 \in \mathbf{T}$, and $-1 < \delta < 1$, let $w(\zeta) = |\zeta - \zeta_0|^\delta$. Then the equality $\|H\| = \cot \frac{\pi(1-|\delta|)}{4}$ was obtained by Krupnik and Verbitsky [18]. Hence $\|P_+\| = \sec \frac{\pi\delta}{2}$.

For complex constants a, b , the formula of the operator norm of $aP_+ + bP_-$ on $L^2(wdm)$ was obtained by Feldman, Krupnik and Markus (cf. [11], [13, Section 13.5], [30]) as the following:

$$\|aP_+ + bP_-\| = \sqrt{\gamma + \left(\frac{|a|+|b|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|a|-|b|}{2}\right)^2},$$

where

$$\gamma = \left| \frac{a-b}{2} \right|^2 (\|P_+\|^2 - 1).$$

If $\gamma=0$, then the right hand side equals to $\max(|a|, |b|)$.

For example, suppose $w(\zeta) = |\zeta - 1|^{1/2}$. Then $\|P_+\| = \sec \frac{\pi}{4} = \sqrt{2}$, and $\gamma = \left| \frac{a-b}{2} \right|^2$. Let

$$c = \|aP_+ + bP_-\| = \sqrt{\left| \frac{a-b}{2} \right|^2 + \left(\frac{|a|+|b|}{2} \right)^2} + \sqrt{\left| \frac{a-b}{2} \right|^2 + \left(\frac{||a|-|b||}{2} \right)^2},$$

and let $d = \left| \frac{(a-b)c}{c^2 - a\bar{b}} \right|$. Then $d \leq 1$, and for every distinct complex constants a and b , we have

$d = \frac{1}{\sqrt{2}}$ by the calculation, and $w(\zeta) = |\zeta - 1|^{1/2}$ has the Helson-Szegő representation:

there exist real functions u and v such that $w = \exp(u + \tilde{v} + \text{const.})$ a.e.,
 $\|v\|_\infty \leq \arccos d = \frac{\pi}{4}$ and $|u| \leq \cosh^{-1}\left(\frac{\cos v}{d}\right) = \cosh^{-1}(\sqrt{2} \cos v)$ a.e. on $\mathbf{T}$.

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